Tunneling Resistivity of a One-Dimensional Random Lattice and the Petersburg Problem

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The resistivity of a one-dimensional lattice consisting of randomly distributed conducting and insulating sites is considered. Tunneling resistance of the form $\rho_0 n e^{bn}$ is assumed for a cluster of n adjacent insulating sites. In the thermodynamic limit, the mean resistance per site diverges at the critical filling fraction $p_c = e^{-b}$, while the mean square resistivity fluctuations diverge at the lower filling fraction $p_{c_2} = p_c^2$. Computer simulations of large but finite systems, however, show only a very weak divergence of resistivity at p_c and no divergence of the fluctuations at p_{c_2}. For finite lattices, calculation of the resistivity at the critical filling is shown to be simply related to the Petersburg problem. Analytic expressions for the resistivity and resistivity fluctuations are obtained in agreement with the results of computer simulations.

KEY WORDS: Tunneling; resistivity; Petersburg problem; random lattice; fluctuations; critical filling; mean and most probable values; divergence.

1. INTRODUCTION

A number of one-dimensional random systems display a sensitivity to statistically rare events. This feature may give rise to a situation in which "typical" behavior is very different from what obtains if one takes a simple ensemble average.[3] Since a number of subtle questions then arise, we find it

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[3] For recent examples of this type of behavior, see Refs. 1 and 2.

useful to consider a simple model which exhibits this sensitivity and which allows detailed analysis.

Our model system consists of a linear array of sites on which insulating and metallic units are placed at random. For simplicity, we take the resistance of the metallic sections to be zero, while the "tunneling" resistance of a cluster of n consecutive insulating units, each of unit length, is[4]

$$R_n = \rho_0 n e^{bn} \tag{1}$$

where ρ_0 and b are positive constants.

Consider an ensemble μ of chains, not necessarily distinct, each consisting of N sites. Each site has the probability p of being occupied by an insulating unit, and for simplicity we employ periodic boundary conditions. In the limit that the ensemble μ becomes infinite, the mean number $\langle m(n) \rangle$ of insulating clusters of size n is[5]

$$\langle m(n) \rangle = \begin{cases} N(1-p)^2 p^n & \text{if } 1 \leqslant n < N - 1 \\ N(1-p)p^n & \text{if } n = N - 1 \\ p^n & \text{if } n = N \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

In units of the resistivity ρ_0, the mean resistivity averaged over the ensemble μ is

$$\begin{aligned} \langle \rho \rangle &= \frac{1}{\rho_0 N} \sum_{n=1}^{N} \langle m(n) \rangle R_n \\ &= (1-p)^2 \left[\frac{e^\alpha - (N-1)e^{\alpha(N-1)} + (N-2)e^{\alpha N}}{(1 - e^\alpha)^2} \right] \\ &\quad + (1-p)(N-1)e^{\alpha(N-1)} + e^{\alpha N} \end{aligned} \tag{3}$$

where $\alpha = \ln p + b$. It is worth noting that

$$\langle \rho \rangle = \frac{1}{2N} \frac{\partial}{\partial b} \langle G \rangle \tag{4}$$

where

$$\langle G \rangle = 2 \sum_{n=1}^{N} \langle m(n) \rangle e^{bn} \tag{5}$$

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[4] This relation holds for metal–insulator junctions at low temperatures in certain situations; for example, see Ref. 3.

[5] See for instance Ref. 4.

Thus $\langle G \rangle$ may be viewed as a generating function of the mean resistivity. If we now take the thermodynamic limit $N \to \infty$, we see that there is a critical value p_c of the probability p, given by $p_c = e^{-b}$ (or $\alpha = 0$) below which the resistivity becomes intensive:

$$\lim_{N \to \infty} \langle \rho \rangle = p e^b \left(\frac{1-p}{1-pe^b} \right)^2 \tag{6}$$

At $p = p_c$, $\langle \rho \rangle = \frac{1}{2}(1 - p_c)^2 N^2 + O(N)$, while for $p > p_c$ the mean resistivity grows exponentially with the length of the chain. Thus, in the thermodynamic limit, a conductor-to-insulator transition is expected. Computer simulations show that the resistance per site increases with N for $p \gtrsim p_c$. However, the observed divergence is much weaker than predicted by Eq. (3). The computed value of the resistivity at $p \gtrsim p_c$ for large but finite N is almost always found to be much smaller than the computed mean value from Eq. (3). This discrepancy between the mean resistivity and the values obtained from computer simulations is in close analogy to the Petersburg paradox discussed below.

In the Petersburg game, a single trial consists of tossing a true coin until it falls heads; if this occurs on the rth throw, the player receives 2^r ducats. The game may be amended[5] so that the player receives nothing if a trial takes more than T tosses. The expectation value E of the player's profit from a single trial is

$$E = \frac{2}{2} + \frac{2^2}{2^2} + \frac{2^3}{2^3} + \cdots \frac{2^T}{2^T} = T$$

The paradox is that even for finite (but large) T, the average profit obtained per trial in everyday experience is much less than T. This problem, for $T \to \infty$, was first discussed by Daniel Bernoulli[6] in 1730; it has subsequently been addressed by Cramer, Fontaine, D'Alembert, Buffon, Bequelin, Condorcet, Laplace, Poisson et al.[7]

The problem of tunneling resistivity and the Petersburg problem may be related if each site on the lattice is regarded as corresponding to a toss of the coin. If $p = 0.5$, conducting sites correspond to a true coin falling heads, and a chain of N sites corresponds to a Petersburg game of as many trials as there are conducting sites and total number of N tosses. For simplicity, we have again applied periodic boundary conditions. The mean number of trials yielding 2 ducats (i.e., consisting of 1 throw) is equal to the mean number of conducting sites minus the mean number of insulating clusters. Adding all contributions, we get for the expectation value (ensemble

average) of the profit

$$\langle E \rangle = \left[N(1-p) - N(1-p)p - p^N \right] 2 + \sum_{u=1}^{N} \langle m(n) \rangle 2^{n+1}$$

$$= \tfrac{1}{2} N(N+1) + 2 - 2^{-N+1}$$

For large N,

$$\langle E \rangle \simeq 2 \sum_{n=1}^{N} \langle m(n) \rangle 2^n = \frac{N^2}{2} = \langle G \rangle \big|_{p=p_c}$$

where $\langle G \rangle \big|_{p=p_c}$ is the generating function of Eq. (5) evaluated at the critical value of p ($p_c = \tfrac{1}{2}$ in our case).

Both in the tunneling problem and in the present version of the Petersburg game, the mean resistivity for $p \gtrsim p_c$ and the mean profit in the game are almost always much greater than the outcome of the experiments. A modern treatment of the Petersburg problem is given by Feller.[5] The salient feature is that the dominant contribution to the mean value comes from the improbable occurrence of large clusters of insulating sites (runs of tails). The mean number of such clusters in the ensemble is much less than 1. In any realization, the number of clusters of any size must be an integer. The number of very large clusters will therefore almost always be 0. In order to discuss the expected outcome of computer experiments, we next turn to the evaluation of the most likely outcome of a single experiment, as well as the average resistivity and deviations from this value of a limited size sample. To do this, we propose a method of analysis, which, when applied to the Petersburg problem, gives results in agreement with Feller.[5]

2. THE MOST PROBABLE OUTCOME

If a single chain is selected at random from the ensemble μ of chains of length N, the most probable number of clusters of size n in this chain can be taken to be

$$[m(n)] = \text{Integer}(\langle m(n) \rangle)$$

where [ ] denotes the most probable value and Integer(x) is the integer nearest to x. If M chains are chosen at random from the ensemble, the most probable value of the average number of clusters of size n in the set of M chains is, to a good approximation,

$$[m(n)]_M = \frac{1}{M} \{ \text{Integer}(M\langle m(n) \rangle) \} \tag{7}$$

Since $\langle m(n) \rangle$ is a monotonically decreasing function of n, there exists a most probable maximum cluster size $n_{\max}$ such that $[m(n)]_M = 0$ for all

$n > n_{max}$. We may determine n_{max} by setting $\langle m(n) \rangle = 1/2M$ in Eq. (1), giving

$$n_{max} = \text{Integer}\left\{ \frac{-\ln\left(2NM(1-p)^2\right)}{\ln p} \right\} \tag{8}$$

for $n_{max} < N$; else $n_{max} = N$. An alternative method of arriving at a cutoff is as follows. The mean number of clusters with length n greater than some integer ν is

$$\sum_{n=\nu+1}^{N} \langle m(n) \rangle = N(1-p)^2\left(\frac{p^{\nu+1} - p^{N+1}}{1-p} \right) \simeq N(1-p)p^{\nu+1} \quad \text{for large } N$$

The most probable maximum cluster size in a sample of M chains is estimated by setting this equal to $1/2M$, giving

$$n_{max} = \text{Integer}\left\{ \frac{-\ln\left(2NM(1-p)p\right)}{\ln p} \right\}$$

The difference between these two estimates is only significant if $p \simeq 1$.

Our analysis is illustrated in Fig. 1 by the result of a computer simulation. Chains of length N were sequentially filled with probability p.
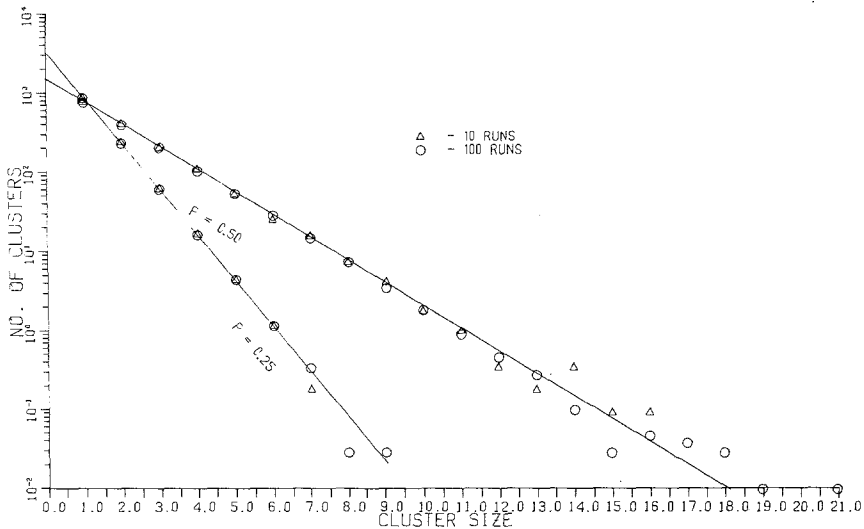

Fig. 1. Average number of clusters versus cluster size for chains of length $N = 10,000$. The solid lines show the mean number of clusters $\langle m(n) \rangle$. Results of computer simulations are denoted by $\bigcirc$ for $M = 100$ and $\triangle$ for $M = 10$. Note that for $p = 0.5$, simulations give $n_{max} = 21$ and 16, respectively, for $M = 100$ and 10, while Eq. (8) gives 19 and 16. For $p = 0.25$, simulations give $n_{max} = 9$ and 7 while theory gives 10 and 8.

The average number of clusters of a given size was obtained by averaging over M such runs. The predicted mean number $\langle m(n) \rangle$ of clusters is also shown.

For ease of calculation, it is useful to approximate the most probable average number of clusters $[m(n)]_M$ in a sample of M chains by allowing $M[m(n)]$ to assume noninteger values below the cutoff, i.e.,

$$[m(n)]_M = \begin{cases} \langle m(n) \rangle, & \text{for } n \leqslant n_{\max} \\ 0, & \text{otherwise} \end{cases}$$

The most probable value of the average resistivity from M simulations is then

$$[\rho]_M = \frac{1}{\rho_0 N} \sum_{n=1}^{N} [m(n)]_M R_n \simeq \frac{1}{\rho_0 N} \sum_{n=1}^{n_{\max}} \langle m(n) \rangle R_n$$

$$= (1-p)^2 \left\{ \frac{e^\alpha - (n_{\max}+1)e^{\alpha(n_{\max}+1)} + n_{\max}e^{\alpha(n_{\max}+2)}}{(1-e^\alpha)^2} \right\} \qquad (9)$$

At the critical filling p_c, $\alpha = 0$, both the numerator and denominator vanish, but $[\rho]_M$ has the limit

$$[\rho]_M \big|_{p=p_c} = \tfrac{1}{2}(1-p_c)^2 n_{\max}(n_{\max}+1) \qquad (10)$$

$[\rho]_M$ is a mostly smooth function of p with small jumps each time $n_{\max}$ increases by 1. At the critical filling, we have

$$\frac{[\rho]_M}{\langle \rho \rangle}\bigg|_{p=p_c} \simeq \frac{1}{N^2} \ln^2(NM) \qquad (11)$$

In the thermodynamic limit $N \to \infty$ there is still a resistivity divergence in $[\rho_M]|_{p=p_c} \sim \ln^2 N$, but this divergence is much weaker than that of the mean resistivity and would be difficult to detect for chains with large but finite N. For $p > p_c$, the most probable resistivity of Eq. (9) increases approximately linearly with N, while the mean resistivity of the ensemble increases exponentially with N. The results of computer simulations are expected to approach the mean value only if the right-hand side of Eq. (11) equals 1, that is, if $M \sim e^{N - \ln N}$. The signature of the behavior near $p = p_c$ is therefore the rather curious fact that the average resistivity of M samples tends to increase with M. In Fig. 2 we show the results of computer simulations which illustrate this behavior.
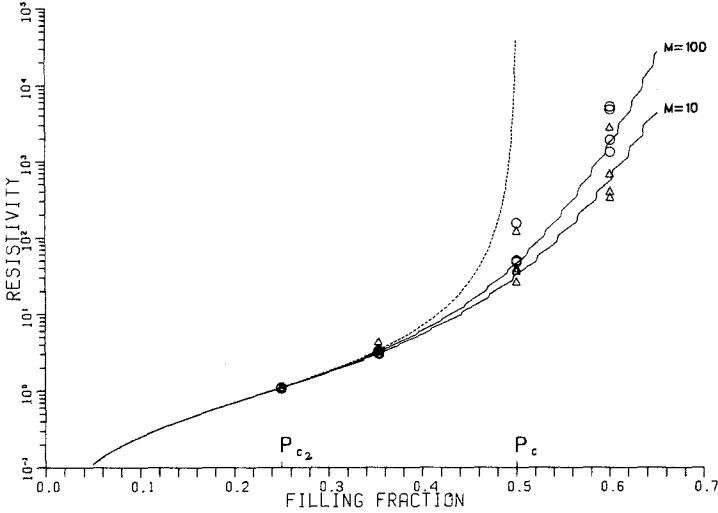
Fig. 2. Average resistivity of M samples versus filling fraction for chains of length $N = 10$, 000 and $b = \ln 2$. The dashed line shows the mean resistivity $\langle \rho \rangle$ in the ensemble μ from Eq. (3), while the solid lines show the most probable average resistivity $[\rho]_M$ of M samples from Eq. (9). Results of computer simulations are denoted by $\bigcirc$ for $M = 100$ and $\triangle$ for $M = 10$.

In a single chain, if $b = \ln 2$, the most probable value of the quantity G at the critical filling $p_c = \frac{1}{2}$ is given by

$$[G]\big|_{p=p_c} = 2 \sum_{n=1}^{n_{max}} \langle m(n) \rangle 2^n = \frac{N}{2} n_{max} = \frac{N}{2} \frac{\ln(N/2)}{\ln 2} \qquad (12)$$

For large N, the most probable profit $[E]$ from a Petersburg game of N tosses is $(N/2)(\ln(N/2)/\ln 2)$. Alternatively, the expected number of trials η in a game of N tosses is $\eta = N/2$. This gives $[E] = \eta \ln \eta / \ln 2$ in agreement with Feller.[5]

3. FLUCTUATIONS

In addition to the most probable resistivity, it is interesting to examine the resistivity fluctuations, i.e., the root mean square deviation of the resistivity from the mean defined by

$$\langle \chi \rangle^2 = \langle \rho^2 \rangle - \langle \rho \rangle^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\langle m(i)m(j) \rangle - \langle m(i) \rangle \langle m(j) \rangle \right) \frac{R_i R_j}{\rho_0^2 N^2} \qquad (13)$$

It is straightforward to show that

$$
\langle m(i)m(j)\rangle = \begin{cases} Np^{i}(1-p)^{2}\{\delta_{ij}+p^{j}(1-p)^{2}(N-i-j-3) \\ \qquad\qquad +2p^{j}(1-p)\} & \text{if } i+j+2 < N \\[6pt] Np^{i}(1-p)^{2}\{\delta_{ij}+p^{j}\} & \text{if } i+j+2 = N \\[6pt] Np^{i}(1-p)^{2} & \text{if } i=j \text{ and} \\ & \qquad \dfrac{N}{2} \leqslant i < N-1 \\[6pt] Np^{i}(1-p) & \text{if } i=j=N-1 \\[6pt] p^{i} & \text{if } i=j=N \\[6pt] 0 & \text{otherwise} \end{cases}
$$

$$(14)$$

The mean fluctuation $\langle\chi\rangle$ can be exactly evaluated using Eq. (14) and Eq. (1). For $0 < p < p_{c}$ and large N, keeping only dominant terms, this gives

$$
\langle\chi\rangle^{2} \simeq \frac{1}{N}\frac{(1-p)^{2}}{(1-e^{\gamma})^{3}}\left\{e^{\gamma}+e^{2\gamma}-(N+1)^{2}e^{\gamma(N+1)}+(2N^{2}+2N-1)e^{\gamma(N+2)}\right.
$$
$$
\left.-N^{2}e^{\gamma(N+3)}\right\}
$$

where $\gamma = \ln p + 2b$. There is a critical value of the probability p given by $p_{c_{2}} = e^{-2b} = p_{c}^{2}$ below which the mean square fluctuations go to zero as $1/N$ in the thermodynamic limit $N \to \infty$; for $p \ll e^{-2b}$, $\langle\chi\rangle^{2} \simeq pe^{2b}/N$. As the critical filling $p_{c_{2}}$ ($\gamma = 0$) is approached, the fluctuations tend to the limiting value

$$
\langle\chi\rangle^{2}\big|_{p=p_{c_{2}}} = \tfrac{1}{3}(1-p)^{2}N^{2}+O(N)
$$

and above this critical filling $\langle\chi\rangle^{2}$ diverges exponentially with N. Again, this divergence is not seen in computer simulations; the discrepancy, as before, is due to the difference between the most probable value and the mean value in the ensemble.

If M chains are chosen at random from the ensemble μ, the most probable average value of the product $m(i)m(j)$ is of the order

$$
\left[m(i)m(j)\right]_{M} = \frac{1}{M}\left\{\text{Integer}(M\langle m(i)m(j)\rangle)\right\}
$$

The region of the i–j plane outside which $\langle m(i)m(j)\rangle \leqslant 1/2M$ can be obtained from Eq. (14); this gives $i+j \leqslant 2n_{\max}$ if $n_{\max}$, from Eq. (8), is less than $(N-1)/2$. Since most of the contributions to $\langle\chi\rangle^{2}$ come from terms where $i \approx j$, we assume that this region can be approximated by $1 \leqslant i$

$\leqslant n_{max}$ and $1 \leqslant j \leqslant n_{max}$, and that $M[m(i)m(j)]$ can assume noninteger values below the cutoff. Thus

$$\left[m(i)m(j)\right]_M = \begin{cases} \langle m(i)m(j)\rangle, & \text{if } 1 \leqslant i \leqslant n_{max} \text{ and } 1 \leqslant j \leqslant n_{max} \\ 0, & \text{otherwise} \end{cases}$$

The most probable resistivity fluctuations $[\chi]_M$ in a set of M chains is then given by

$$[\chi]_M^2 = \sum_{i=1}^{n_{max}} \sum_{j=1}^{n_{max}} \{\langle m(i)m(j)\rangle - \langle m(i)\rangle\langle m(j)\rangle\} \frac{R_i R_j}{\rho_0^2 N^2} \qquad (15)$$

The results of simulations and the most probable fluctuations $[\chi]_M$ are shown in Fig. 3 together with the mean fluctuations $\langle \chi \rangle$ in the ensemble. As in the case of resistivity, the most probable fluctuations for $p \gtrsim p_{c_2}$ increase with the number of samples M. The most probable fluctuation at $p = p_{c_2}$ is

$$[\chi]_M^2 \simeq \frac{1}{3} \frac{(1-p)^2}{N} \left\{ \frac{-\ln 2MN(1-p)^2}{\ln p} \right\}^3$$

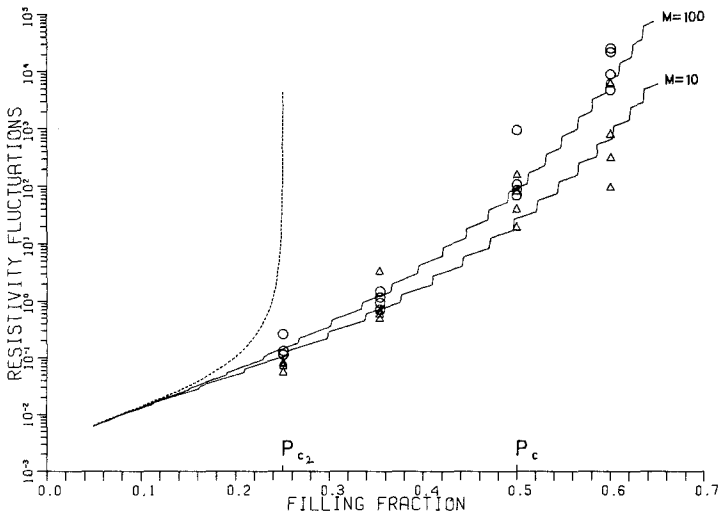and for $p < p_c$, the most probable fluctuations decrease with chain length

Fig. 3. Resistivity fluctuations of M samples versus filling fraction for chains of length $N = 10,000$ and $b = \ln 2$. The dashed line shows the mean fluctuation $\langle \chi \rangle$ in the ensemble μ from Eq. (13), while the solid lines show the most probable fluctuation $[\chi]_M$ in M samples from Eq. (15). Results of computer simulations are denoted by $\bigcirc$ for $M = 100$ and $\triangle$ for $M = 10$.
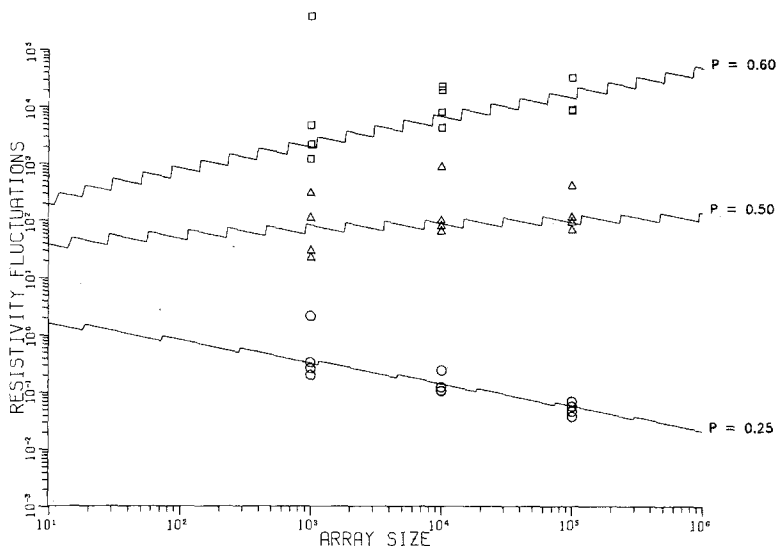
Fig. 4.   Resistivity fluctuations versus chain length N; $b = \ln 2$ and $M = 100$. The solid lines show the most probable fluctuations $[\chi]_M$ from Eq. (15). Results of computer simulations are denoted by O for $p = p_{c_2} = 0.25$, $\triangle$ for $p = p_c = 0.5$, and $\square$ for $p = 0.6$.

N. Thus, if the sample size M is fixed, there is no evidence of divergence in $[\chi]_M$ for $p = p_{c_2}$. This behavior is illustrated in Fig. 4, where we show the most probable fluctuations for $p = p_{c_2}$, $p = p_c$, and $p > p_c$ together with results of computer simulations. The curious behavior that the fluctuations decrease with N for $p_{c_2} < p < p_c$ while they increase with N for $p > p_c$ can be understood from the following crude argument.

   If we assume that the sum of the resistivities in a set of M chains is dominated by the resistance of the largest occurring cluster of size $n_{\max}$, then this sum $\rho_{\mathrm{tot}} \simeq n_{\max} e^{b n_{\max}}/N$ and $[\chi]_M^2 = \hat\rho_{\mathrm{tot}}^2(1/M - 1/M^2)$. Substituting for $n_{\max}$ from Eq. (8) gives

$$\ln[\chi] = \left(-\frac{b}{\ln p} - 1\right)\ln N + O(\ln\ln N)$$

Thus the slope of the line $\ln[\chi]$ vs. $\ln N$ changes sign at $b = -\ln p$, or $p = p_c = e^{-b}$, and the most probable fluctuations are expected to decrease with N if $p < p_c$ and increase if $p > p_c$.

4.   DISCUSSION

   In conclusion, we have found that the tunneling resistivity in a randomly filled chain is sensitively dependent on the improbable occur-

rence of large insulating clusters. We have established a close analogy with the Petersburg game where expected profits sensitively depend on the improbable occurrence of large runs of tails in a coin-tossing experiment. In a set of M samples of finite size, both the average resistivity and the resistivity fluctuations are significantly different from their mean values in the ensemble μ of chains for filling fractions near and above the critical value. Theory is provided to describe these properties in good agreement with computer experiments.

In the present theory, there is a logarithmic divergence with N of the most probable resistivity at $p = p_c$; the mean resistivity in the ensemble μ diverges as N^2 at this same critical filling. The most probable fluctuations also diverge logarithmically at $p = p_c$; however, the mean fluctuations in the ensemble μ diverge at the lower critical filling $p_{c_2} = p_c^2$. One might attempt to see the divergence in the most probable resistivity by plotting $\ln[\rho]$ vs. $\ln N$; however, another test of the present theory is the prediction illustrated in Fig. 4 that the most probable fluctuations decrease with N for $p < p_c$ and increase with N for $p > p_c$. An interesting feature of this model is the prediction that the most probable average resistivity of M samples tends to increase with the number of samples M for filling fractions near and above the critical value. (In simulations, however, this increase is very irregular as expected from the theory.) Other quantities, such as the geometric mean resistivity and conductivity of a single chain or a bundle of chains can also be studied. Computer simulations indicate that above the critical filling the geometric mean increases with increasing N and both the average conductivity and the conductivity of a bundle will go to zero, but, as expected, with much smaller fluctuations. Detailed studies of these quantities will be published elsewhere.

ACKNOWLEDGMENTS

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